High School Students' Understanding and Retention of the Connection between Two Ways of Constructing an Ellipse

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Abstract

Conic section is the study of various types of shapes resulted from cutting cones with planes. Another way to construct these shapes is to consider loci of points under certain conditions. An ellipse is one type of conic sections. Here, the cutting plane is flatter than the surface of the cone. The foci of the resulting ellipse are the points where the plane touches Dandelin's spheres—the spheres that fit inside the cone. An ellipse is also the locus of points whose sums of distances to the two foci are constant. One of the problems of learning conic section is the lack of connection between the two constructions. Therefore, we constructed a set of concrete manipulatives based on Dandelin's spheres to enhance high school students' understanding and retention of the connection between the two ways of constructing an ellipse. Four high school students participated in this study wherein learning process proceeded through group discussion. They explored the manipulatives by following a guided worksheet. After one month, we assessed their retention by using a semi-structured interview whose result was analyzed to extract common themes. The result showed that two students were able to see the connection between the two ellipse constructions while the others could not.

Keywords: *Dandelin's spheres, Ellipse, Concrete manipulative, Retention, Cutting cone, Conic section*

1. Introduction

Conic section is one of the high school mathematics contents. It refers to the curves generated by cutting a conical surface with a flat plane, resulting in four types of curves: circle, ellipse (Figure 1), parabola, and hyperbola. These shapes can also be constructed by considering the loci of points under certain conditions as shown in Figure 2. The third way to define a conic section is to use an algebraic equation as shown in Figure 3. In traditional classrooms, teachers give some algebraic problems related to these algebraic definitions, and students spend hours working on the problems (Salinas & Pulido, 2016). Students find difficulty in working with the algebraic definition, its equations, the reflection properties of standard conic section and the complexity of the numerous new terms, which are the cause of students ability to integrate the concepts and apply into a real-life situation (Chen, 2013). The problem of learning conic section is the absence of the explanations of the connections among the three constructions. Students cannot visualize how cutting a cone is connected to the loci and to its algebraic definition. The common features of all conic sections are there foci and directrices, which are the keys to making connections between the first two ways of construction. To be able to visualize the foci after cutting cones, Dandelin's spheres can be used. Apostal (1961) stated that using Dandelin's spheres in the case of an ellipse could produce what could be recognized as a proof by a picture that a cone cut obliquely by a plane resulted in an ellipse as defined by its focal property (the locus-of-point construction). Even though the proof is in the form of a picture, it might be hard for high school students to understand. Visualizing the three-dimensional image of the cone and the intersecting plane from those on a screen is a difficult task for students. So, a hands-on model of the cone and the cutting plane should be able to help students create a reasonable mental model. Students would be able to see all the components of the conic section. Students should not have the mental maturity to grasp abstract mathematical concepts presented in words or symbols alone; they need many experiences with concrete materials and drawings for learning to occur. The use of the concrete manipulatives makes sense for an abstract topic in mathematics. Concrete manipulatives can provide a bridge between the concrete and abstract levels of many mathematical topics. It can also help to retain knowledge.

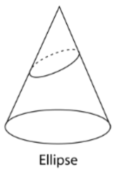

Figure 1 Cutting cone

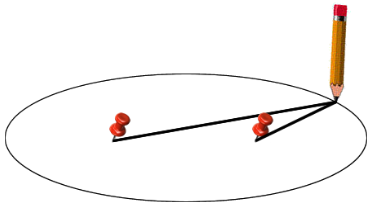

Figure 2 Ellipse's definition

$Circle: (x-h)^2 + (y-k)^2 = r^2$

$Ellipse: \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

$Parabola: (y-k)^2 = 4p(x-h) \;\; or \;\; (x-h)^2 = 4p(y-k)$

$Hyperbola: \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \;\; or \;\; \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

Figure 3 Conic algebraic equations

2. Objectives

2.1 Research Objectives

2.1.1 To construct students' understanding of the connection between two ways of constructing an ellipse.

2.1.2 To examine students' retention in constructing the connection between two ways of constructing an ellipse.

2.2 Research Questions

2.2.1 What are the students' understanding of the connection between two ways of constructing an ellipse?

2.2.2 What is the students' retention in the connection between two ways of constructing an ellipse?

3. Materials and Methods

3.1 Research Design

Based on the research objectives, concrete manipulatives were developed to support students' learning process. The intervention was designed to support the first objective, which is to enhance students' understanding of the connection between those two constructions of an ellipse. The semi-structured interview was designed to examine students' retention. To address the research questions, the case study approach was utilized (Figure 4).

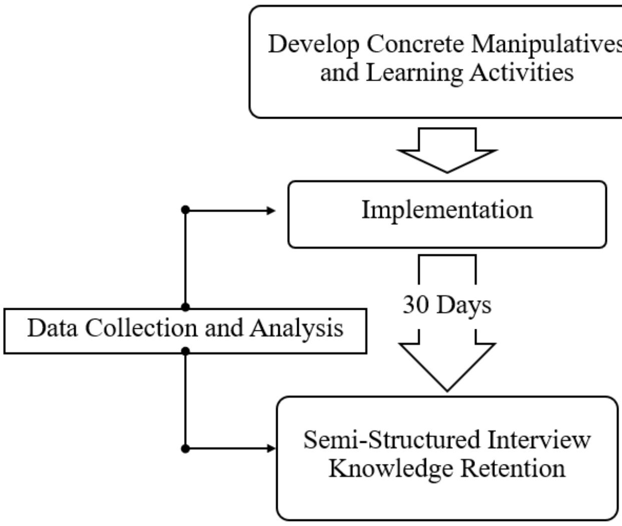

Figure 4 scheme of research method

3.2 Sample/Target group

Four female high school students (grade 10) participated in the activities. They were attending a school in a suburb of Bangkok, Thailand. They have never studied the topic of the conic section before participating in the activities.

3.3 Research Instrument

3.3.1 Concrete manipulatives

A set of concrete manipulatives based on Dandelin's spheres is used to enhance students' understanding and retention of the connection between two ways of constructing an ellipse. It consists of a metallic conic frame (Figure 5), a metallic circle which is a special case of an ellipse and a handful of metallic ellipses corresponding to selected pairs Dandelin's spheres (Figure 6), and a handful of spheres to be used as Dandelin's spheres (Figure 7).

Figure 5 Metallic conic frame

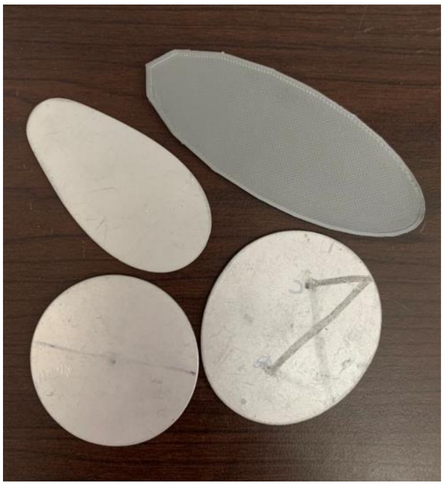

Figure 6 Metallic circle and ellipse

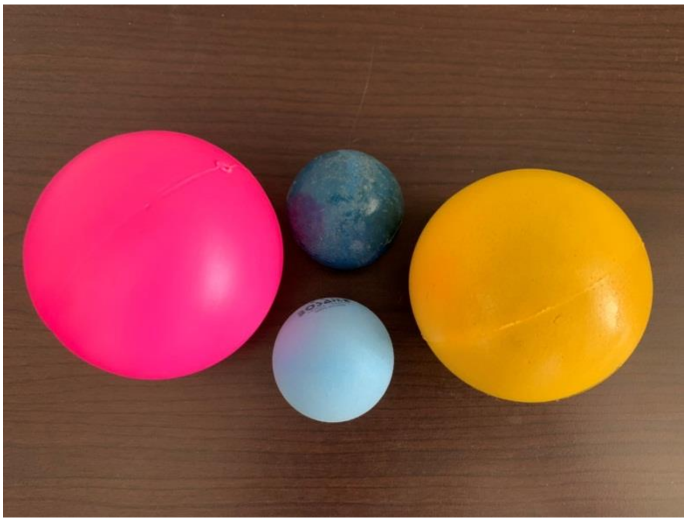

Figure 7 Handful of spheres

3.3.2 Guided worksheet

A guided worksheet was then used to guide students through the process of ellipse construction. In detail, the worksheet consisted of the ellipse's locus-of-point definition, the property of a pair of lines tangential to circle originating from the same point outside the circle, and the step-by-step illustrative guide to locating the foci and relating them to those in the locus-of-point definition.

3.3.3 Semi-structured interview

The questions in the semi-structured interview were designed to motivate students to self-correct and extend or generalize their understanding.

3.4 Data Collection

The intervention was divided into two sessions. The first session comprised three stages of group activities while each student's knowledge retention was checked in the second session a month later.

The first stage of the first session aimed to help students to know what a conic section is, the meaning of the center of a circle and its relationship to the focus points of an ellipse, and the ellipse's locus-of-point definition. The stage began by asking the students about some of the mathematical terms that would be used during the activities. Two spheres that touch each other after fitting into the metallic cone were used to show that they touched the metallic circle at the center of the circle which resulted from cutting the cone horizontally (Figure 8). Then students were asked to constructed an ellipse from the locus-of-point definition using a pair of pins, a pencil, and a string (Figure 2).

Figure 8 Circle from two spheres touched

The second stage aimed to help students identify the foci of an ellipse resulted from cutting the cone by using concrete manipulative based on Dandelin's spheres and relate the foci to those in the locus-of-point definition. After the first stage, the students learned to demonstrate the use of the spheres and the cone by showing how to construct a circle. They could observe the way the construction worked. By putting their hands on the concrete manipulative, the students should be able to store their experiences and corresponding meanings in the long term memory, which could be used in their future work.

The third stage aimed to summarize what had been learned from the previous stages. The students were asked to conclude their understanding by themselves by answering open-ended and close-ended questions. Even the close-ended questions were not always simple. Every question required students to think more than one step.

The second session aimed to assess each student's retention of the concept of an ellipse resulted from cutting a cone. A semi-structured interview was used to examine their retention. To determine how they thought, every question was open-ended. The questions were designed to motivate them to self-correct and extend or generalize their understanding.

In all sessions, students were asked for the permission of voice record of their performance.

3.5 Data Analysis

For data analysis, the interactions between the students and the researcher during the activities and the semi-structured interview were transcribed. The transcripts were extracted to find common themes and analyzed to see how much the students understood and retained their knowledge.

4. Results

The results were in the form of transcriptions. The four students will, henceforth, be referred as S1–S4 while the researcher will be referred as Res.

4.1 Results from the first session.

Among the four students, only two (S1 and S2) participated actively in the activities. S3 just answered shortly, kept smiling, and tried to follow her friends' answers. S4 did not answer any questions; she just sat and listened to her friends' answers.

4.1.1 Results from the first stage: constructing an ellipse from its locus-of-point definition.

Res: *I am going to use the same string to construct another ellipse by changing the positions of the fixed points.* (Res indicated the positions of the new fixed points by spreading her hands further apart.) *What would the resulting ellipses look like?*
S1: *It will be smaller than the previous one.*
Res: *What does smaller mean? Is it more eccentric? What is the cause of it?*
S2: *Because of the movement of the fixed points.*
S1: *Two fixed points are further apart. That is why the ellipse is more eccentric.*

Both students responded to all questions. They constructed an ellipse from the given condition.

4.1.2 Results from the second stage: relating an ellipse resulted from cutting cone to that constructed in the first stage.

Res: *From point A* (indicating a point on the perimeter of the ellipse in the cone), *is there any straight-line tangent to the big* [Dandelin's] *sphere which lies on the conic surface?*
S1: *Um …*
S2: *This line* (pointing her finger at the line).
S1: *From point A, go up this way to the big sphere* (pointing her finger at the same line as that pointed by S2).
Res: *Let's name that point.*
S2: *Ok, let's call it point D.*
S1: *Okay.*
Res: *So, what is the name of this line?*
S1: *AD* (writing it down on the guided worksheet).
Res: *Then, is there any straight-line from point B* (where the big Dandelin's sphere touched the ellipse) *to the ellipse's perimeter equal to the previous line?*
S1: *Um …*
Res: *Now we have line AD. So, what line is equal to AD and still on the ellipse's surface?*
S2: *Line AB*
S1: (writing line AB on the model)
Res: *So, line AB is equal to line AD. Let's see next question. What is the sum of distances from point A to point B and point A to point C* (where the small Dandelin's sphere touched the ellipse) *without using numbers?*

S2: *AC plus AB equal to AD … Um … no … Is it related to its angle?*
Res: *What is the angle of your meaning? Where is the angle you refer to?*
S2: *Um … it is not about the angle.*
Res: *Let's see what line we have now. Now we have two pairs of lines which are equal in length. And by considering the ellipse's definition.*
S1: *Um … AC plus AB equal to BC.*
S2: *Really?*
Res: *Let's draw the line that you are thinking it is equal to.*
S1: (drawing the line on the model) … *Oh! No! there is no line BC and it is not equal to.* (So far, four lines had been drawn: AB, AC, AD, and AE, the one from point A to the small Dandelin's sphere which lay on the conic surface.)
Res: *Think about how many lines we have now. Now we have line AB, AC, AE and AD.*
S2: *I think the sum of AB and AC is equal to this line* (pointing her finger at the line on the conic surface between the two spheres).
Res: *Yes, it is.*
S1: (writing AB + AC = ED)
Res: *What about if I change the position of point A to another point on ellipse's perimeter, called point G, is there any change? Does the sum of distances change? What do you think? What should it be?*
S3: *Not change.*
S2: *It will be the same length as ED.*
S1: *Equal.*
Res: *Can you describe the sum of these distances without using the name of any point?*
S2: *Equal.*
S1: *Um … all equal.*
Res: *Turn back to the ellipse's definition.*
S1: *Constant!*
Res: *Yes, that's right.*

The students could not follow the guided worksheet by themselves. The researcher needed to interactively guide until they were able to see the connection between the two ways of constructing an ellipse.

4.1.3 Results from the third stage: summarizing the knowledge.

Res: *Can you conclude the relations between the ellipse resulted from cutting cone and that from its definition?*
S2: *The relations between two ways of construction are two fixed points and the tangent of the spheres and the plane.*
S1: *Two fixed point are the pins.*

Both students tried to summarize their knowledge. They helped each other to conclude in their own words.

4.2 Results from the second session.

Based on their answers in the semi-structured interview, S1 and S2 could retain the knowledge of both ways of constructing an ellipse. They could explain their understanding in sentences with correct keywords confidently. S3 could retain just some keywords and explained without confident. S4 did not retain the knowledge of constructing an ellipse.

4.2.1 S1's interview.

Res: *Why do you think where the* [Dandelin's] *sphere touches the* [elliptical] *plane is a fixed point? Can you explain?*
S1: *Um … Yes, sure. It starts with the sphere fitting the plane and cone. Then draw a line from the point on the ellipse's perimeter to the touching point of the small sphere and the plane.*

It is the same length with this line (pointing her finger at the line on the surface of the cone).

Res: Then ...

S1: Then, these two lines are equal in length.

Res: What about the touching point of the big [Dandelin's] *sphere and the plane?*

S1: We should draw a line from the touching point of the big sphere and the plane to the previous point on the ellipse's perimeter. This line is equal in length to the line up to the big sphere (trying to point her finger on the model).

Res: And ...

S1: The sum of the distances between the fixed points and the ellipse's perimeter is equal to the distance between the tangent circles of the two spheres.

Res: Is it possible to cut the cone and get the shape of an oval (egg-like)*?*

S1: No, it is impossible.

Res: Why do you think it is impossible?

S1: Because when we slice the cone, we can get an ellipse in which the sum of distances is constant, the same as its definition. So, it is impossible to get an oval shape.

S1 showed that she could connect between the ellipse resulted from cutting a cone and that from the locus-of-point definition. She knew that the ellipse is symmetrical from its definition. The sum of distances is always constant for an ellipse. An oval is not symmetrical.

4.2.2 S2's interview.

Res: Why do you think where the [Dandelin's] *sphere touches the* [elliptical] *plane is a fixed point? Can you explain?*

S2: There are two fixed points. The sum of this line (pointing her finger at the line from the point that the small sphere touched the ellipse to its perimeter) *and the other line* (pointing her finger at the line from the same point on the ellipse's perimeter to the point that the big sphere touched the ellipse) *is equal to the distance between the two spheres.*

Res: Can you show the distance between the two spheres?

S2: Um ... This line, the distance from the tangent of small sphere and the cone to the tangent of big sphere and the cone.

Res: Is the distance between the two spheres constant?

S2: Yes, but it is constant only for this ellipse. (She answered with confident)

Res: Is it possible to cut the cone and get the shape of an oval?

S2: Um ... No.

Res: Why?

S2: It does not make sense to get an oval from cutting the cone.

S2 also showed that she could see the connections between the two constructions of an ellipse.

5. Discussion

From the results of the first stage, when the researcher proposed the new positions of the fixed points, S1 and S2 could accurately predict the result with correct explanation, indicating that, by manually constructing an ellipse, they understood the effect of the fixed points on the resulting ellipse.

From the results of the second stage, the students could not follow the guided worksheet by themselves. Students randomly answered and the researcher needed to repeat the questions and gave them more guidance. They may lack the ability to see two-dimensional geometric relationships on the three-dimensional model (hands-on model). Even with the concrete manipulatives and guided worksheet, the geometric relationships underlying the connection were too difficult for them to see.

From the results of the third stage, S1 and S2 helped each other to conclude their understanding. Both students could use keywords to explain the connection between the two constructions of an ellipse, even though they could not finish the previous stage without guidance. The concrete nature of the activity must have helped them to construct their schema of the geometric relationships. Thus, they could relate between the two fixed points and the sum of the distances from these two fixed points to a point on the perimeter of the two ways of constructing an ellipse.

A month later, the interview of S1 showed that she still retained most of her knowledge about the connection between these two constructions of an ellipse. She could explain the relationship between those two constructions. Moreover, she could explain why it was impossible to cut the cone and get the shape of an oval without hesitation, indicating that she understood the nature of an ellipse. From the interview of S2, in the process of constructing an ellipse from cutting a cone, she could explain without reluctant. However, despite knowing that cutting a cone could not result in an oval, she failed to provide any explanation. Their answers regarding an oval reflected different levels of understanding of the nature of an ellipse. During the activities, S1 was answering questions instantly and interacting with the hands-on model the most, while S2 was answering questions without holding the hands-on model. This showed that physically manipulating the hands-on model could enhance the learning better than observing someone else doing it. Thus, there should be enough the hands-on models for every student.

6. Conclusion

This study identified how the students understood and retained their knowledge of the connection between two ways of constructing an ellipse. The results showed that only two students understood the two ways of constructing an ellipse and could retain their knowledge. The other two students had some difficulty in constructing knowledge, which affected their knowledge retention. In further research, we will reconstruct the guided worksheet to be more suitable for weaker students.

7. Acknowledgements

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