A determination formula on the copula-based estimation of Value at Risk for the portfolio problem

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Abstract

Value at Risk (VaR) is one of the widely employed risk measures in quantitative risk management. Because of its readiness of use, both theoretical and practical researches have been extensively made so far. Here, in this paper, we are concerned with the estimation of VaR for the portfolio problem; the portfolio consists of two random variables. Our innovative point is that we do not necessarily assume the independence between random variables but the possibility of nonlinear relation. To analyze the nonlinear dependence, a copula function method is customarily used; as is well-known, copula provides a flexible tool for treating the possible nonlinear relation. We derive the determination formula of VaR analytically in the case of Archimedean copulas, which may be served as elementary machinery of computation. Empirical studies are also implemented for the problem of estimating VaR in stock indexes. The results show that our obtained formula works rather well.

Keywords: *Value at Risk (VaR), Portfolio problem, Copulas, Archimedean copulas, Determination formula*

1. Introduction

In our modern globally-connected financial systems, various kinds of crises, from small to a large extent, are frequently observed these days. To name a few, the subprime mortgage crisis in the USA around 2008, the European debt crisis since late 2009, and many others. In order to manage these instabilities, financial institutions should monitor possible overall risks continuously, and therefore it is desirable to utilize certain characteristic quantity which represents the total risk involved.

Value at Risk (VaR) is introduced in this respect. Because of its simplicity and readiness of use, VaR is now well recognized as one of the principal risk measures in financial risk management. A theoretical study, as well as various practical estimation methods, have been investigated, and much progress has been made so far. For further information and background materials of VaR, we refer, for instance, to Duffie and Pan (1997) and/or McNeil, Frey & Embrechts (2005) and the references therein.

In this paper, we are concerned with the estimation of VaR for the portfolio problem. The portfolio we consider is formulated as the linear combination of two random variables. It is noted that VaR is commonly defined for a single random variable. An innovative point of our research is that these two random variables are not necessarily assumed to be independent, but they have a possible nonlinear relation. Although the assumption of independence is very stylized in financial analysis, it is believed that many phenomena show some dependence features.

To deal with nonlinear dependence, a copula function method is customarily used; because of its flexibleness, copulas provide handy tools for the analysis. For background materials about copulas, we refer, for instance, to Joe (1997), Frees and Valdes (1998), Nelsen (2006) and the references therein. We also refer to the recent work of Yoshiba (2018) for another aspect of copulas.

The rest of the paper is organized as follows. In Section 2, the objectives of our research are settled. Section 3 provides preliminary materials. We state our main result in Section 4, which followed by empirical studies in Section 5. Section 6 concludes.

2. Objectives

In order to develop a better procedure for estimating VaR, the objectives of our study are now settled as follows.

We establish a new analytical formula to evaluate VaR for the portfolio problem, in the case that nonlinear dependence is assumed to be described by Archimedean copula.

We implement empirical studies with real data of the S&P 500 and Jakarta Stock Exchange so that we compare the effectiveness of copula-based estimation with a standard method.

To be specific further, we briefly present our analytical result a little in details. Below, technical ingredients and background information will be explained in later sections.

Let X and Y denote random variables, whose joint distribution function H is represented by a copula C; namely,

$$H(x,y) = P(X \leq x, Y \leq y) = C\big(F_X(x), F_Y(y)\big)\,,$$

where $F_X(x) = P(X \leq x)$ and $F_Y(y) = P(Y \leq y)$ denote marginal distribution functions of X and Y, respectively. We consider the portfolio represented by a random variable Z of the form

$$Z = \lambda X + (1-\lambda)Y \qquad (0 < \lambda < 1)\,,$$

and its Value at Risk

$$\mathrm{VaR}_\beta(Z) = F_Z^{(-1)}(\beta) = \inf\{t \mid F_Z(t) \geq \beta\} \qquad (0 < \beta < 1)\,.$$

Then, under the assumption that C is an Archimedean copula with generator φ and the condition that F_Z is continuous and strictly monotone for simplicity, we establish that $\mathrm{VaR}_\beta(Z)$ is evaluated as the solution of the equation, whose proof is given in Section 4.

$$\beta = \frac{d}{dz}\int_0^z C_\varphi\left(F_X\left(\frac{1}{\lambda}(z-y)\right), F_Y\left(\frac{y}{1-\lambda}\right)\right)dy.$$

We exhibit empirical studies on estimating VaR in Section 5, which follows our previous work Molina Barreto and Ishimura (2020).

3. Materials and Background Issues

3.1 Value at Risk

We first recall the definition of Value at Risk (VaR) for completeness of our presentation.

Let X be a random risk variable and denote by $F_X(x) = P(X \leq x)$ its distribution function. VaR at the confidence level β $(0 \leq \beta < 1)$ is then simply defined by

$$\mathrm{VaR}_\beta(X) := F_X^{(-1)}(\beta) = \inf\{t \mid F_X(t) \geq \beta\}.$$

Because of its simplicity and readiness for use, VaR now becomes a standard benchmark of risk factors. We mention that, however, VaR is defined for a single random variable; on the other hand, we here deal with VaR for the portfolio problem, which involves two random factors, and we note that the treatment in the case of the portfolio problem seems to be not so well investigated despite its importance in the study of risk management.

3.2 Copula

Next, we recall the definition of the copula in the case of the bivariate joint distribution.

Definition. A function C defined on $I^2 := [0,1] \times [0,1]$ and valued in I is called a copula if the following conditions are fulfilled.

(i) For every $(u,v) \in I^2$,

$$C(u,0) = C(0,v) = 0, \quad \text{and} \quad C(u,1) = u \ C(1,v) = v.$$

(ii) For every $(u_i, v_i) \in I^2$ $(i = 1,2)$ with $u_1 \leq u_2$ and $v_1 \leq v_2$,

$$C(u_1, v_1) - C(u_1, v_2) - C(u_2, v_1) + C(u_2, v_2) \geq 0$$

The requirement (ii) is referred to as *the 2-increasing condition*. We also note that a copula is continuous by its definition.

The well-known result due to Sklar (1973), who employed the term "copula" almost for the first time, gives the basic property of copulas. We here recall Sklar's theorem in bivariate case, for completeness of our presentation.

Theorem. (Sklar's theorem) *Let H be a bivariate joint distribution function with marginal distribution functions F and G; that is,*

$$\lim_{x\to\infty} H(x,y) = G(y),$$
$$\lim_{y\to\infty} H(x,y) = F(x).$$

Then there exists a copula, which is uniquely determined on $RanF \times RanG$, such that

$$H(x,y) = C\big(F(x), G(y)\big).$$

Conversely, if C is a copula and F and G are distribution functions, then the function H defined above is a bivariate joint distribution function with margins F and G.

3.3 Archimedean copula

An important class of copulas is given by the so-called Archimedean copulas. We recall for completeness which is the Archimedean copulas. Let φ be a convex function defined on I and valued in $[0,\infty]$ such that φ is strictly decreasing and verifies $\varphi(1) = 0$. Let denote by $\varphi^{[-1]}$ the pseudo-inverse of φ; that is, Dom $\varphi^{[-1]} = [0,\infty]$, Ran $\varphi^{[-1]} = I$, and

$$\varphi^{[-1]} = \begin{cases} \varphi^{(-1)}(t) & \big(0 \leq t \leq \varphi(0)\big) \\ 0 & \big(\varphi(0) \leq t \leq \infty\big) \end{cases}.$$

It is then possible to prove that the function C defined on I^2 by

$$C(u,v) = \varphi^{[-1]}(\varphi(u) + \varphi(v))$$

satisfies the properties (i)(ii) in Definition above, and thus C provides a copula.

Copulas of the form above are called Archimedean copulas and the function φ is called a generator of the copula. The class of Archimedean copula gives a wide range of applications. For example, the generator $\varphi(t) = -\log\big(1 - t^\theta\big)$ $(\theta \geq 1)$ yields the copula $C(u,v) = 1 - \big((1-u)^\theta + (1-v)^\theta - (1-u)^\theta (1-v)^\theta\big)^{\frac{1}{\theta}}$. This family of the copula is studied by Joe (See Joe (1997)). Other examples of Archimedean copulas include:

Clayton copula:

$$\varphi(t) = \theta^{-1}\big(t^{-\theta} - 1\big) \quad (\theta \in [-1,\infty) \setminus \{0\}), \qquad C(u,v) = \big(\max\{u^{-\theta} + v^{-\theta} - 1, 0\}\big)^{-\frac{1}{\theta}}.$$

Gumbel copula:

$$\varphi(t) = (\log t)^{\theta} \qquad (\theta \geq 1), \qquad C(u,v) = \exp(-[(-\log u)^{\theta} + (-\log v)^{\theta}]^{1/\theta}).$$

Ali-Mikhail-Haq copula:

$$\varphi(t) = \log\frac{1-\theta(1-t)}{t} \qquad (-1 \leq \theta < 1), \qquad C(u,v) = \frac{uv}{1-\theta(1-u)(1-v)}.$$

Frank copula:

$$\varphi(t) = -\log\frac{e^{-\theta t}-1}{e^{-\theta}-1} \qquad (\theta \in (-\infty,\infty)\setminus\{0\}), \qquad C(u,v) = -\frac{1}{\theta}\log\left(1+\frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1}\right).$$

There are many other Archimedean copulas. For more details, we refer to an excellent book by Nelsen (2006).

4. Results and Discussion

Now we state our main observation in this article.

Theorem. (Determination formula for $\mathrm{VaR}_{\beta}(Z)$) *Suppose that* X, Y *be nonnegative random variables, whose joint distribution function* H *is represented by an Archimedean copula* C_{φ} *with generator* φ*; namely,*

$$H(x,y) = P(X \leq x, Y \leq y) = C_{\varphi}(F_X(x), F_Y(y)),$$

where $F_X(x) = P(X \leq x), F_Y(y) = P(Y \leq y)$ *are marginal distribution functions of* X, Y*, respectively. Let* $Z = \lambda X + (1-\lambda)Y$ $(0 < \lambda < 1)$ *be a portfolio. Then, its Value at Risk at the confidence level* $(0 < \beta < 1)$, *that is,* $\mathrm{VaR}_{\beta}(Z) = F_Z^{(-1)}(\beta) = \inf\{t \mid F_Z(t) \geq \beta\}$ *can be attained as the solution* z *of the equation ;*

$$\beta = \frac{d}{dx}\int_0^x C_{\varphi}\left(F_X\left(\frac{1}{\lambda}(x-y)\right), F_Y\left(\frac{y}{1-\lambda}\right)\right)dy.$$

In particular, we have

$$\beta = \frac{d}{dx}\int_0^x C_{\varphi}\left(F_X\left(\frac{1}{\lambda}(x-y)\right), F_Y\left(\frac{y}{1-\lambda}\right)\right)dy\bigg|_{x=\mathrm{VaR}_{\beta}(Z)}$$

Proof for simplicity, we assume that $F_Z(z)$ is continuous and strictly monotone. General cases are treated with obvious modifications. First, we see that $\mathrm{VaR}_{\beta}(Z)$ is determined by the equation;

$$\beta = P(Z \leq z) = P(\lambda X + (1-\lambda)Y \leq z)$$

$$= \int_0^z ds \int_0^s \frac{\partial^2 C_{\varphi}}{\partial u \partial v}\left(F_X\left(\frac{t}{\lambda}\right), F_Y\left(\frac{s-t}{1-\lambda}\right)\right)\frac{1}{\lambda}f_X\left(\frac{t}{\lambda}\right)\frac{1}{1-\lambda}f_Y\left(\frac{s-t}{1-\lambda}\right)dt$$

$$= \int_0^{\frac{z}{1-\lambda}}\left(\int_0^{\frac{z}{\lambda}-\frac{1-\lambda}{\lambda}y} \frac{\partial^2 C_{\varphi}}{\partial u \partial v}(F_X(x), F_Y(y))f_X(x)\,dx\right)f_Y(y)dy$$

$$= \int_0^{\frac{z}{1-\lambda}} \frac{\partial C_{\varphi}}{\partial v}\left(F_X\left(\frac{z}{\lambda}-\frac{1-\lambda}{\lambda}y\right), F_Y(y)\right)f_Y(y)dy$$

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$$= \int_0^{\frac{z}{1-\lambda}} \left[\frac{\partial}{\partial y}\left(C_\varphi\left(F_X\left(\frac{z}{\lambda}-\frac{1-\lambda}{\lambda}y\right), F_Y(y)\right)\right) + \frac{1-\lambda}{\lambda}\frac{\partial C_\varphi}{\partial u}\left(F_X\left(\frac{z}{\lambda}-\frac{1-\lambda}{\lambda}y\right), F_Y(y)\right) f_X\left(\frac{z}{\lambda}-\frac{1-\lambda}{\lambda}y\right)\right] dy.$$

Now, thanks to the assumption that C_φ is Archimedean, we derive

$$P(\lambda X + (1-\lambda)Y \le z)$$

$$= \int_0^{\frac{z}{1-\lambda}} \frac{1-\lambda}{\lambda} \frac{\varphi'\left(F_X\left(\frac{z}{\lambda}-\frac{1-\lambda}{\lambda}y\right)\right)}{\varphi'\left(\varphi^{[-1]}\left(\varphi\left(F_X\left(\frac{z}{\lambda}-\frac{1-\lambda}{\lambda}y\right)\right)+\varphi\left(F_Y(y)\right)\right)\right)} f_X\left(\frac{z}{\lambda}-\frac{1-\lambda}{\lambda}y\right) dy$$

$$= (1-\lambda)\int_0^{\frac{z}{1-\lambda}} \frac{d}{dz}\varphi^{[-1]}\left(\varphi\left(F_X\left(\frac{z}{\lambda}-\frac{1-\lambda}{\lambda}y\right)\right)+\varphi\left(F_Y(y)\right)\right) dy$$

$$= (1-\lambda)\frac{d}{dz}\int_0^{\frac{z}{1-\lambda}} C_\varphi\left(F_X\left(\frac{z}{\lambda}-\frac{1-\lambda}{\lambda}y\right), F_Y(y)\right) dy$$

$$= \frac{d}{dx}\int_0^{x} C_\varphi\left(F_X\left(\frac{1}{\lambda}(x-y)\right), F_Y\left(\frac{y}{1-\lambda}\right)\right) dy\,.$$

This completes the proof.

In the above computation, we note that the boundary conditions of copulas are taken into account. We remark that the original version of the equation has already been employed by Fantazzini (2008) for numerical computation. We have developed a substantially simple formula in the case of Archimedean copulas, which seems to be new in the literature. We also note that our previous result of Molina Barreto, Ishimura, and Yoshizawa (2019) contains an error, and the correct formula is expressed here. For another attempt in the same line of research, we refer to our recent work of Molina Barreto and Ishimura (2020).

5. Empirical Studies

We present an application for the formula to estimate Value at Risk via copula. This empirical analysis is based on our previous research Molina Barreto and Ishimura (2020), where the modelling is done with a Monte Carlo approach for sampling the quantile of the distribution given by the copula.

5.1. Data description

We consider a portfolio composed of two assets: the S&P 500 and Jakarta Stock Exchange Composite Index (JCI). The data contains 2377 daily closing prices from December 7th 2009 to December 6th 2019, and we compute the daily log-returns and ignore the entries that are not available at the same time in any of both markets. The data period excludes the direct effect of the United States subprime mortgage crisis started from 2007. The data is taken from Yahoo Finance, and the implementation is performed with MATLAB.

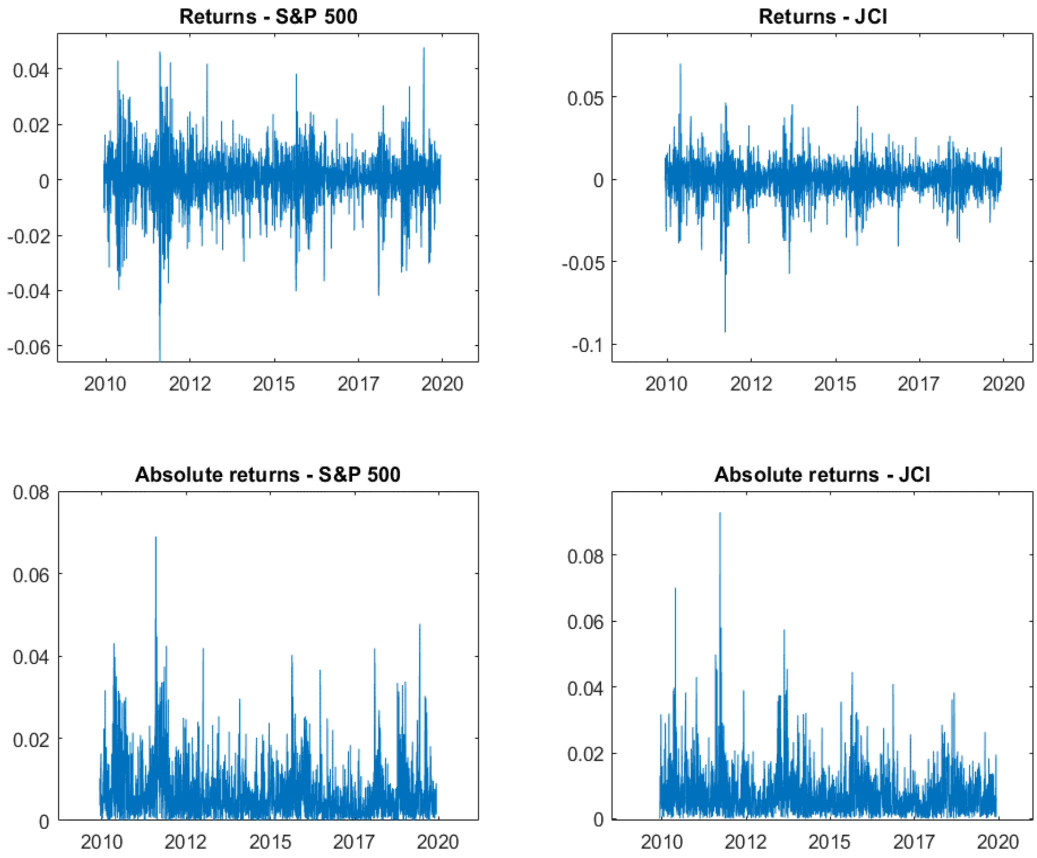

Figure 1 Daily returns and absolute returns of S&P500 and JCI stock indices.

Figure 1 shows plots for both log-returns and table 1 presents the main statistics. We remark the excess of kurtosis, and negative asymmetry is significant in this case.

5.2 Marginal models and Copula estimation

For each marginal series, we consider a general AR(1)-GARCH(1,1) model with innovations with two compounded Gaussian mixture distributions. This idea seems accurate due to the characteristics that can be seen in the series as asymmetry and excess of kurtosis. We have observed that the ARMA-GARCH with normal mixture distributed innovation models fit this kind of series better than plain ARMA-GARCH with normal or t distributions.

For specifying a model for each series we consider ARMA(p,q)-GARCH(r, s) model for asset returns r_t $(t = 1,2, \ldots, T)$ is given by

$$r_t = a_0 + \sum_{i=1}^{p} a_i r_{t-i} + \varepsilon_t + \sum_{j=1}^{q} b_j \varepsilon_{t-j}\,, \qquad \varepsilon_t = z_t \sigma_t\,,$$

$$\sigma_t^2 = c_0 + \sum_{i=1}^{r} c_i \varepsilon_{t-i}^2 + \sum_{j=1}^{s} d_j \sigma_{t-j}^2\,.$$

Here, z_t $(t = 1,2,\dots,T)$ is a sequence of independently distributed (i.i.d.) random variables with K component Gaussian mixture density defined as

$$f_\eta(y) = \sum_{i=1}^{K} \pi_i f(y;\mu_i,\sigma_i) \quad \text{with} \quad f(y;\mu_i,\sigma_i) = \frac{1}{\sqrt{2\pi}\sigma_i}\exp\left\{-\frac{1}{2}\left(\frac{y-\mu_i}{\sigma_i}\right)^2\right\}.$$

Table 1 Descriptive statistics of daily log-returns of S&P500 and JCI stock indices.

Statistics	S&P 500	JCI
Mean	0.000441	0.000384
Standard Deviation	0.009583	0.010504
Minimum	-0.068958	-0.092997
Median	0.000611	0.001002
Maximum	0.047775	0.070136
Kurtosis	7.381000	9.112100
Asymmetry	-0.461540	-0.591570

For the estimation of the parameters, we use the Inference Function for Margins (IFM) method, to be precise, we first compute the estimator of the parameters for the ARMA-GARCH and normal mixture distribution with the quasi-likelihood estimation process and transform the series into uniform one with its cumulative distributions function. Once these parameters have been calculated, the next step is to estimate the parameters for the copula. Figure 2 shows the plot for conditional variance and standardized residuals with the estimated parameters.

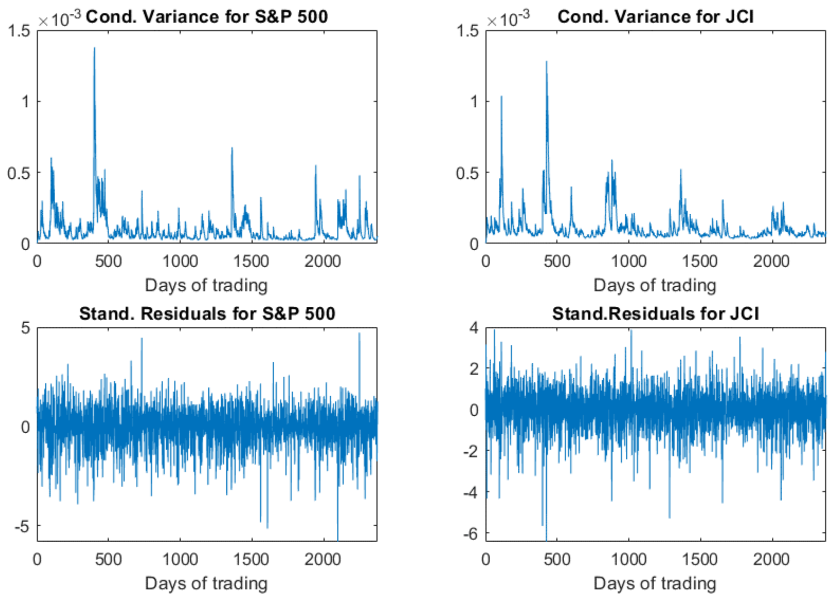

Figure 2 Conditional variance and Standardized Residuals for log-returns series of S&P500 and JCI stock indices.

This selection is performed to see that there is no autocorrelation nor squared autocorrelation in the residuals. We also performed Ljung Box test to infer that the null hypothesis is not rejected from lag 1 to 5. Values for Kolmogorov-Smirnov (KS), Chi-Square Goodness of t-test (CSG) and Anderson-Darling test used for the uniformity test for the standardized residuals are all accepted for the significance level of 95%.

We have implemented this methodology with Clayton, Gumbel and Frank copula. Other examples of copulas are also possible to integrate to this estimation. Once the data is transformed into uniform data by its estimated cumulative distribution function, we construct the likelihood function for the copula and seek for the parameters that maximize it.

5.3. Estimation of Value at Risk

We again consider the portfolio of equal weight. First, we estimate the parameters using the data from $t = 1$ to $t = 1376$ as the initial window and update the parameters each day as for the marginal distributions as for the copula. Our target is to find the solution formula for VaR at the level $\beta = 0.05$ and $\beta = 0.01$ concerning the data from $t = 1377$ to $t = 2376$ (1000 days). We then compare the forecast VaR with the actual return of the portfolio.

By looking at the value of violations, we could infer that the performance of the proposed model is better over classical estimations of VaR as Historical or Variance-Covariance, thanks to the effect of nonlinear dependence given by the copula, as well as the improvement of implementing it to the computation of VaR. We also compared with benchmark models like Variance-Covariance. In all cases, the model with Clayton-Normal mixture gives the best results. Data is exhibited in figure 3 and figure 4.

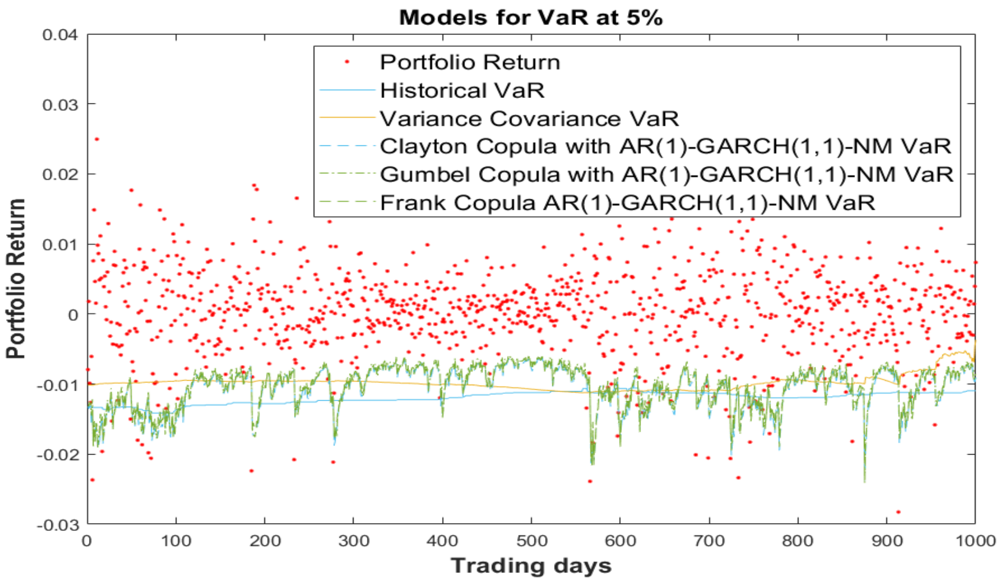

Figure 3 Estimation of Value at Risk for significance level at 5% for the portfolio with several methods.

5.4 Backtesting

To ascertain the outcome of computation, several back-testing methodologies are considered. We have used the Binomial test (Bin), Kupiec's POF test (POF), and Christoffersen's test (CCI), respectively. See Christoffersen (1998). The result is shown in table 2. Our empirical analysis has shown that the proposed models with copulas result in better estimations than models such as Historical or Variance-Covariance methods. Thanks to the property of copula, we can explain a better nonlinear correlation between the two indexes studied here. In effect, for extreme losses, the copulas (except for Gumbel copula) give better estimates and pass all the three backtests. We can also observe similar behavior for the three copulas.

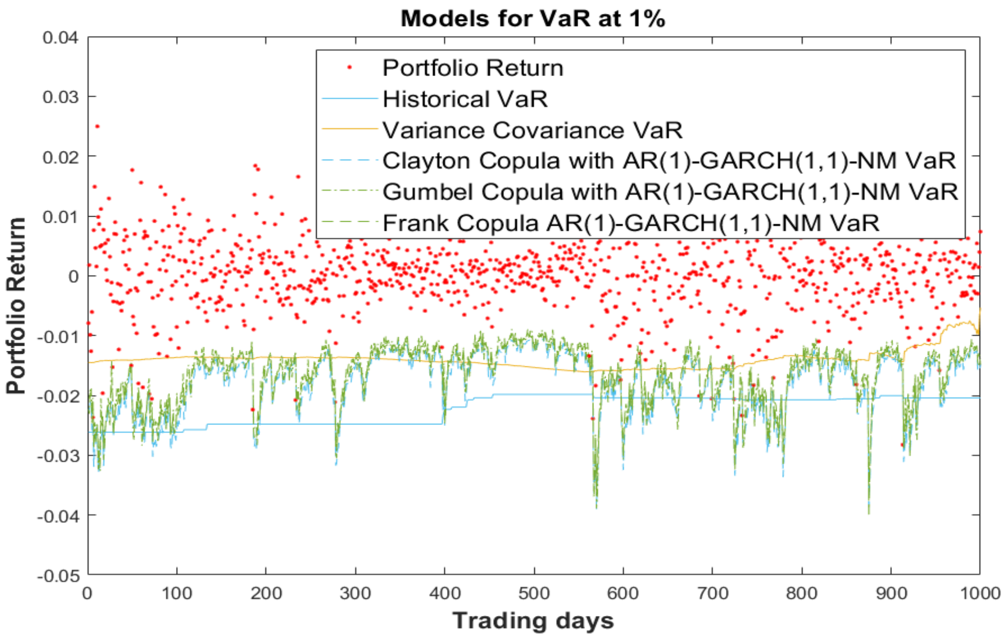

Figure 4 Estimation of Value at Risk for significance level at 1% for the portfolio with several methods.

Table 2 Results of the VaR backtesting for the models with the significance level $\beta = 5\%$ and $\beta = 1\%$. Test conducted here includes the Binomial test, Proportion of Failures test (POF), Conditional Coverage Independence test (CCT). Each test has its p-value for the significance at the 95% level.

Model	Binomial test	Z-score	p-value	Failures	Prop. Failures	POF test	Likelihood ratio	p-value	CCI test	Likelihood ratio	p-value
	b=5%			50	0.05						
Clayton NM-5	accept	-0.72548	0.4682	45	0.045	accept	0.54382	0.46085	accept	3.5136	0.06087
Gumbel NM-5	accept	-0.43529	0.6634	47	0.047	accept	0.19318	0.66029	reject	5.0767	0.02425
Frank NM-5	accept	-0.72548	0.4682	45	0.045	accept	0.54382	0.46085	accept	1.6801	0.19491
Historical-5	accept	-1.59600	0.1105	39	0.039	accept	2.74690	0.09744	accept	3.1141	0.07762
Var. Covar-5	accept	1.74110	0.0817	62	0.062	accept	2.82600	0.09275	accept	2.4326	0.11884
	beta=1%			10	0.01						
Clayton NM-1	accept	0.31782	0.7506	11	0.011	accept	0.09783	0.75444	accept	0.2449	0.62066
Gumbel NM-1	reject	2.22470	0.0261	17	0.017	reject	4.09100	0.04311	accept	0.5886	0.44295
Frank NM-1	accept	1.58910	0.1120	15	0.015	accept	2.18920	0.13898	accept	0.4573	0.49887
Historical-1	accept	-1.58910	0.1120	5	0.005	accept	3.09370	0.07859	accept	0.0503	0.82254
Var. Covar-1	reject	4.44950	0.0861	24	0.024	reject	14.22100	0.00016	accept	1.1817	0.27702

6. Conclusion

The estimate of Value at Risk (VaR) for the portfolio problem is discussed. The portfolio consists of two risk factors, which are not necessarily independent but possibly nonlinearly related. We further assume that these random effects are represented by an Archimedean copula. In this setting, we have established the determination formula for evaluating the VaR of the portfolio. The formula is simple and may become a basis for further investigation. We also believe there are fruitful applications both in theoretical and practical fields.

In our previous research of Molina Barreto and Ishimura (2020), it is indicated that a copula-based approach yields a better estimate of VaR compared to standard ARMA-GARCH modelling; precisely stated, obtained values of copula-based estimation, in a sense, reflect the total risk of the portfolio effectively. Our formula may be served as a handy tool for further study on VaR estimation. The current empirical studies also show the effectiveness of copula-based methods. However, more deep investigations should be performed in order to conclude what method is appropriate. We continue our researches on VaR estimation.

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