Development of A Calculus Lesson Plan Using the CRA Approach Combined with Scaffolding Techniques

Poohrich Siriputcharoen, Mingkhuan Phaksunchai and Chalita Toopsuwan*

Faculty of Science, King Mongkut's University of Technology Thonburi,
126 Pracha Utit Road, Bang Mot, Thung Khru, Bangkok, Thailand
*Corresponding author, E-mail: mingkhuan.pha@kmutt.ac.th

Abstract

This paper aims to achieve two objectives: 1) to design and develop a calculus lesson plan for high school students using the Concrete-Representational-Abstract (CRA) approach combined with scaffolding techniques and 2) to assess the quality of the above-mentioned CRA-based lesson plan. The CRA approach, integrated with scaffolding techniques, serves as the foundation for designing the lesson plan. It consists of three stages: Concrete (learning through real objects or real-life experiences), Representational (learning through representations such as graphs), and Abstract (learning through abstract mathematical concepts). Scaffolding questions were incorporated into each stage to guide students' thinking and facilitate self-directed learning. The lesson plan focusing on high school calculus topics were divided into four separated plans: a plan addressing the limit of a function, a plan on the continuity of a function at points, a plan on the continuity of a function on intervals, and a plan on the slope of a curve. The quality of each plan was assessed by three experts using a Likert scale assessment form (ranging from 1 to 5). The assessment results showed that all four lesson plans achieved average scores between 4.00 and 5.00, reflecting their "Very good" quality. This research can serve as a foundation for further research on adaptive teaching methods, potentially influencing how educators develop and implement lesson plans to improve student learning outcomes in mathematics.

Keywords: *The Concrete-Representational-Abstract (CRA) Approach, Scaffolding Technique, Calculus*

1. Introduction

The academic achievement of students in engineering, science, and technology programs at the university level is significantly influenced by their grasp of fundamental calculus concepts. A strong foundation in these areas not only enhances critical thinking skills but also prepares students for the complexities of real-world applications in their respective fields. Developing a strong understanding of calculus concepts is crucial for cultivating the limited supply of future scientists, technologists, mathematicians, and engineers (Roble, 2017; Sadler & Sonnert, 2018).

Despite the acknowledged significance of calculus, research has consistently demonstrated that students often struggle to develop a precise understanding of calculus concepts (Muzangwa & Chifamba, 2012). Classroom instruction tends to prioritize teaching rules and procedures, leading many students to merely perform these actions without truly grasping the underlying concepts (Berry & Nyman, 2003; Bezuidenhout, 2001). Therefore, educators should prioritize effective teaching strategies and resources to enhance students' foundational understanding of calculus concepts.

In mathematics education, various pedagogical approaches have been implemented to enhance students' understanding of calculus concepts. Traditional-Based Learning (TBL) primarily emphasizes direct instruction and procedural fluency, allowing students to develop basic computational skills but often leading to superficial comprehension of abstract mathematical ideas (Khalaf & Zin, 2018). In contrast, Inquiry-Based Learning (IBL) and Problem-Based Learning (PBL) focus on student-centered exploration and real-world problem-solving, promoting deeper conceptual understanding and critical thinking (Khasawneh et al., 2023; Pogorelova, 2023). However, while IBL and PBL have been effective in fostering engagement, they may not provide sufficient support for students struggling with abstract calculus concepts, potentially leading to cognitive overload and difficulty in applying mathematical principles systematically.

According to Tarmizi (2010), students often encounter significant challenges in learning calculus, primarily due to difficulties in visualizing and understanding abstract mathematical concepts. This study which investigated students' performance in solving calculus problems revealed that many students struggled with problem representation, which is crucial for effective problem-solving. The study emphasized that good problem solvers generally construct a representation of the problem to facilitate understanding, highlighting a common obstacle among students in grasping calculus concepts.

The Concrete-Representational-Abstract (CRA) approach, based on Bruner (1966), consists of three stages; Concrete (learning through real objects or real-life experience); Representational (learning through representation); and Abstract (learning through abstract concepts). The CRA focuses primarily on the transition from concrete (manipulatives or physical objects) to abstract representations (symbols or equations), with the representational stage bridging the two. It emphasizes the progression from hands-on experiences to more abstract conceptual understanding. To facilitate the comprehension of mathematical concepts, the CRA approach is generally employed.

According to Wood et al. (1976), Scaffolding is defined as the process that allows a child or beginner to solve a problem, complete a task, or reach a goal that would be beyond their unaided capabilities. Scaffolding techniques significantly enhance students' mathematical conceptual understanding by providing structured support that aids in the grasping of abstract concepts and the development of problem-solving skills. According to Anda and Aman (2022), Frederick and Courtney (2014), and Dy and Lapinid (2023), scaffolding strategies, such as guided questioning, modeling, and providing hints or examples, lead to significant improvements in students' ability to comprehend and apply mathematical concepts. Studies have shown that scaffolding not only improves conceptual understanding but also helps students in overcoming learning challenges, thereby fostering deeper engagement with mathematics.

During the period from 2018 to 2022, the CRA approach was used to enhance mathematics learning outcomes, students' abilities, and students' affective abilities. It was primarily implemented at the elementary level. The implementation of the CRA approach at the junior high school level has been carried out, but it is still minimal (Azzumar & Juandi, 2023) despite research's suggestion of using the CRA approach to teach complex concepts of mathematics in high school. Moreover, according to the mathematics education curriculum outlined by the Ministry of Education in Singapore (2012), the teacher's role is described as that of a facilitator who guides students through the levels of concrete, pictorial, and abstract understanding by offering suitable scaffolding and feedback. Guiding students through the use of scaffolding techniques is necessary to facilitate a deeper understanding of mathematical concepts.

This paper outlines the process of developing a calculus lesson plan using the Concrete-Representational-Abstract (CRA) approach, incorporating appropriate scaffolding to support high school students. Four lesson plans were created, each addressing the concepts of limits of a function, continuity at a point, continuity on an interval, and the slope of a curve.

2. Objectives

The objectives of this research are as follows:

1) To design and develop a calculus lesson plan for high school students using the CRA approach combined with scaffolding techniques.
2) To assess the quality of the aforementioned CRA-based lesson plan.

3. Materials and Methods

In this section, the process of designing a calculus lesson plan was described. Figure 1 show the three-step procedure for creating the lesson plan.

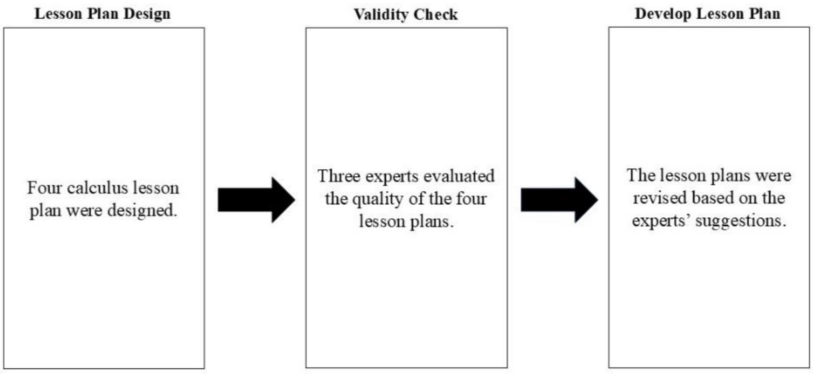

Figure 1 Three-step procedure for creating the calculus lesson plan

The process began with designing the four calculus lesson plans by using the CRA approach combined with scaffolding technique.

Next, the lesson plans were validated by three experts, each selected based on their distinct expertise and qualifications. The first expert is a professor specializing in mathematics education at a prominent university in Thailand, with extensive experience in developing and implementing innovative approaches to mathematics instruction. The second expert is an award-winning high school mathematics teacher recognized nationally for excellence in teaching and effective use of instructional media. The third expert is a seasoned calculus instructor with extensive experience of teaching high school students at leading institutions, known for her proficiency in organizing and delivering calculus instruction effectively.

The materials used in this research consisted of four lesson plan assessment forms. The first assessment form was used by the experts to assess the quality of the lesson plan on the topic of limits of functions. The second form was for evaluating the lesson plan on the topic of continuity of functions at points. The third form was prepared to assess the quality of the lesson plan on the continuity of functions on an interval, while the final form was for evaluating the lesson plan on the topic of the slope of a curve. All lesson plan assessment forms use a Likert scale ranging from 1 to 5. In each assessment form, the criteria used to assess the quality of the lesson plan were based on three main factors. The first criterion was the components of the lesson plan. The second criterion pertained to the organization of teaching and learning activities. The final criterion related to an inclusion of appropriate media and assessment components. An example of the assessment form is presented in Table 1.

Table 1 Lesson Plan Assessment Form

No.	**Assessment List**	**Assessment Level**				
		1	**2**	**3**	**4**	**5**
	Components of the lesson plan					
1	The title of the lesson plan includes comprehensive teaching information, such as the name of the plan, the learning unit title, the grade level, the duration of the lesson, etc.					
2	The standards, indicators, and learning outcomes are clearly and comprehensively specified.					
3	Learning objectives are clearly identified and aligned with the specified indicator standards.					
4	Learning objectives can be measured and evaluated.					
5	The main topic is clearly defined, ensuring that readers can easily understand the purpose of the lesson.					
6	The content is aligned with the indicators, learning outcomes, and learning objectives.					

Table 1 Lesson Plan Assessment Form (Continued)

No.	Assessment List	Assessment Level				
		1	2	3	4	5
	Teaching and organizing learning activities					
7	Learning activities are well-suited to the content and time allocated.					
8	Learning activities are organized in alignment with the learning objectives.					
9	The organization of learning activities in the Concrete stage aligns with the Concrete-Representational-Abstract (CRA) approach, specifically in the Concrete stage.					
10	The organization of learning activities in the Representational stage aligns with the Concrete-Representational-Abstract (CRA) approach, specifically in the Representational stage.					
11	The organization of learning activities in the Abstract stage aligns with the Concrete-Representational-Abstract (CRA) approach, specifically in the Abstract stage.					
12	The organization of learning activities in the Concrete stage can be linked to those in the Abstract stage, with the Representational stage serving as a connecting link.					
13	The questions used to organize learning activities are clear and easy to understand.					
14	The questions used in learning activities effectively guide students in answering questions or solving problems.					
15	The organization of questions in learning activities helps guide students toward the key concept of the lesson					
	Media components and assessment					
16	The media is aligned with the teaching and learning activities.					
17	The assessment tools and methods are aligned with the learning objectives.					
18	The assessment tools and methods are aligned with the learning process and the organization of activities.					
19	The assessment criteria for knowledge, using a 3-point scale, are detailed, consistent, clear, and comprehensive.					
20	The assessment criteria for knowledge, using a 2-point scale, are detailed, consistent, clear, and comprehensive.					
21	The assessment criteria for knowledge, using a 1-point scale, are detailed, consistent, clear, and comprehensive.					
22	The assessment criteria for knowledge, using a 0-point scale, are detailed, consistent, clear, and comprehensive.					
	Total					
	Average total score					

Finally, each lesson plan was revised according to the feedback and suggestions provided by the experts.

3.1 Lesson Plan Design

To design the lesson plans, we began by examining a calculus concept included in the high school curriculum in Thailand. According to the Institute for the Promotion of Teaching Science and Technology (2020), Thailand's mathematics curriculum was revised in 2017. The updated curriculum incorporates calculus at the high school level, categorizing it into three main groups: the first is limit and continuity of a function, the second is derivatives, and the last is integration. In this study, we focus specifically on the concept of limit and continuity of a function, as well as an application of derivatives, particularly the slope of a curve.

To design the calculus lesson plans using the CRA approach combined with scaffolding techniques, the plans should progressively guide students through mathematical concepts starting with concrete examples (e.g., physical models or real-life applications) that illustrate fundamental calculus principles. As students gain familiarity, they should transition to more abstract representations such as graphs, symbols, and formulas, aligning with the principles of CRA-based learning (Bruner, 1966; Leong et al., 2015).

Scaffolding questions would be strategically employed at each stage to probe students' understanding, encourage deeper exploration, and facilitate connections between concrete and abstract concepts. According to Roehler and Cantlon (1997), five distinct scaffolding strategies were frequently employed by educators to facilitate students' acquisition of conceptual understanding. Through observations conducted in two social constructivist classrooms, they highlighted the following primary strategies: offering explanation; inviting student participation; verifying and clarifying student understanding; modeling of desired behaviors; and inviting students to contribute clues. Therefore, the scaffolding questions designed in this study emphasized guiding students to observe familiar situations or representations, such as graphs. Students were encouraged to independently think, explain, and respond to the questions. Following this, the teacher evaluated their answers, addressed any misconceptions, and provided adjustments to align the responses with the intended concept. This approach was grounded in the scaffolding strategies outlined by Roehler and Cantlon (1997).

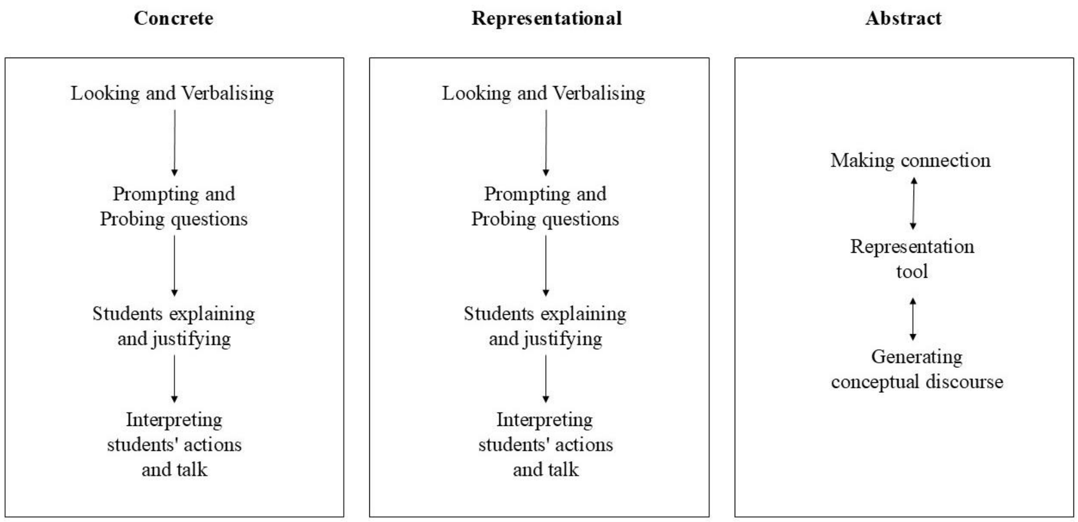

Figure 2 The Concrete-Representational-Abstract Approach Combined with Scaffolding Techniques

Figure 2 shows each stage of the CRA approach in which scaffolding techniques were integrated to guide students in the direction needed for them to grasp the concepts.

3.1.1 Limit of function lesson plan design

The concept of the limit of a function is fundamental in calculus. Specifically, the limit refers to the value that a function $f(x)$ approaches as x gets arbitrarily close to a specific point, either from the left or right. Limits are crucial for defining concepts such as continuity, derivatives, and integrals, serving as the foundation for understanding how functions behave near certain points, even if the function is not explicitly defined at those points. This concept allows mathematicians to handle situations involving infinite values, discontinuities, and infinitesimal changes, which are essential in advanced calculus and real-world applications.

In the first step of designing the limit of a function lesson plan, we began with the Concrete stage. We identified real-life situations that connect to the concept of limits to serve as the Concrete stage in the CRA approach. Specifically, we used the situation of divers exploring the seabed.

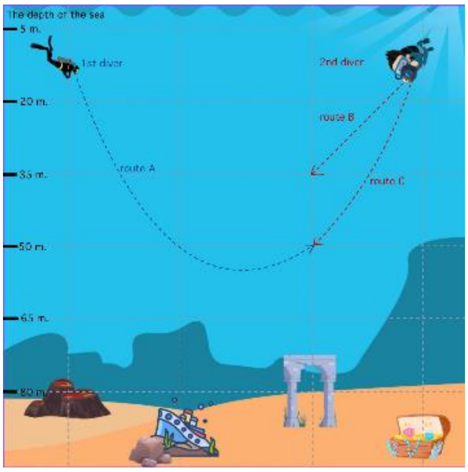

Figure 3 The situation of divers exploring the seabed

From Figure 3, the first and second divers are exploring the sea. The first diver is about to dive in to explore the sea and follows route A. Meanwhile, the second diver is deciding whether to use route B or route C to explore the sea. If we consider the situation from the picture, it can be observed that as the diver moves closer to various positions in the sea, we can examine the depth to which the diver approaches. This concept aligns with the idea of a limit of function. Moreover, in the Concrete stage, we also designed questions to scaffold students. Here is an example of a question: "If the first diver explores the sea and follows route A and the second diver chooses route B, then as both divers move close to the point where the sunken ancient gate is located, at what depth will they be closest to each other?"

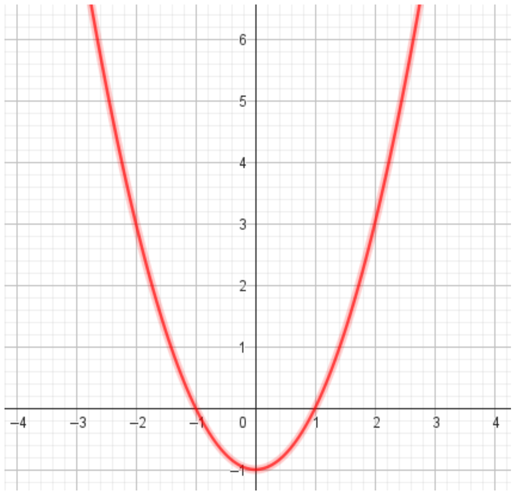

Figure 4 Graphical representation for limit of function

In the next stage, the Representational stage, we aimed to help students connect the concrete scenario to graphical representations and the concept of the limit of a function. The mathematical representations used in this stage included graph of a function, as shown in Figure 4. Additionally, we also designed questions to help students grasp the concept of the limit of a function. For example: "If x approaches from the left-hand side, what value does $f(x)$ approach?". The connection students can make is that as x approaches a specific point, it is represented by the diver moving closer to the position of the sunken object under the sea. Similarly, the value of $f(x)$ corresponds to the depth of the sea.

Finally, in the Abstract stage, we introduced the formal limit notation and helped students connect it to the concrete scenarios and representational tools used in the earlier stages.

3.1.2 Continuity of function lesson plan design

The concept of continuity of a function describes a function that behaves predictably without any abrupt breaks, jumps, or holes in its graph. Formally, a function $f(x)$ is continuous at a point $x = c$ if three conditions are met: $f(c)$ is defined, the limit of function $f(x)$ as x approaches c exists, and the value of the function at $x = c$ is equal to limit of function $f(x)$ as x approaches c. Continuity extends to intervals when a graph of function can be drawn smoothly and continuously within the given interval.

In the initial phase of designing the lesson plan on the continuity of functions, we incorporated a card game that provided a fun, hands-on activity while helping students connect with the concept of continuity in functions.

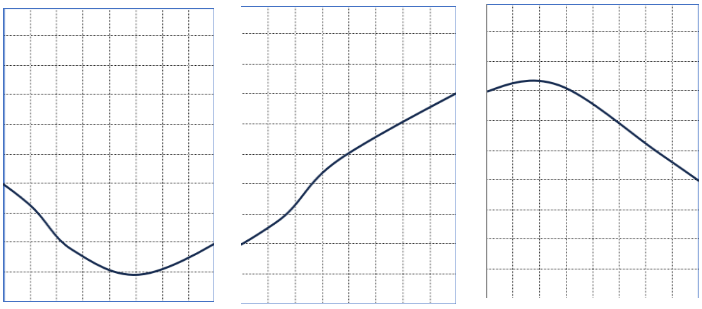

Figure 5 An example of curve cards

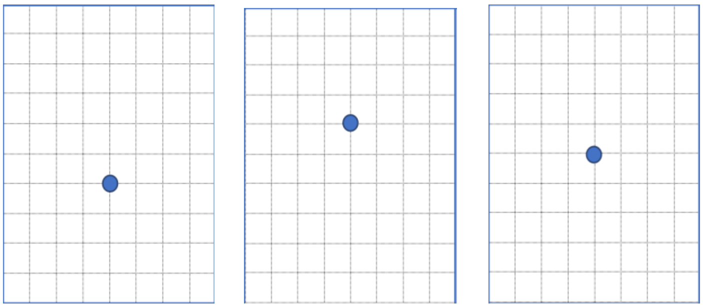

Figure 6 An example of dot cards

Figures 5 and 6 show examples of curve cards and dot cards games used in the Concrete stage. The objective of the games was to create a continuous curve by connecting the curve cards. A dot card must be placed between two curve cards to bridge the connection; otherwise, the curve cards cannot be linked. This game aligns with the concept of continuity in functions. In addition to playing the game, we also posed questions to encourage students to think critically about the continuity of a curve. For example: "What conditions are necessary to connect the curve cards?"

In the Representational stage, our goal was to help students transition from the concrete scenario to graphical representations and the concept of continuity in functions. This stage utilized mathematical representations, such as function graphs, to deepen understanding. Students analyzed these graphs to explore continuity, guided by questions like, “Which function is continuous at the given point?” or “Which function is continuous on the specified interval?”

The connection between the Concrete, Representational, and Abstract stages was established through two key conditions required to link the curve cards. First, the curves on the two cards must align perfectly, representing the idea that the left-hand limit of $f(x)$ equals the right-hand limit of $f(x)$. Second, the points must precisely connect to ensure a seamless transition between the two curves, corresponding to the condition that the value of the function at a given point equals the limit of $f(x)$ as x approaches that point.

3.1.3 Slope of a curve lesson plan design

The slope of a curve at a point, a cornerstone concept in calculus, is defined using the limit of a function and represents the instantaneous rate of change at that point. Unlike a straight line with a constant slope, a curve's slope varies, necessitating a more precise definition. Mathematically, the slope at a point $x = a$ is given by the limit of the difference quotient,

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

where $f(x)$ is the function describing the curve. This formula captures the transition from a secant line, which connects two points on the curve, to the tangent line, which just touches the curve at a single point. As h approaches zero, the secant line's slope converges to that of the tangent line, formalizing the slope's value. This concept underpins the derivative, a critical tool in mathematics for analyzing rates of change, motion, and optimization in diverse applications.

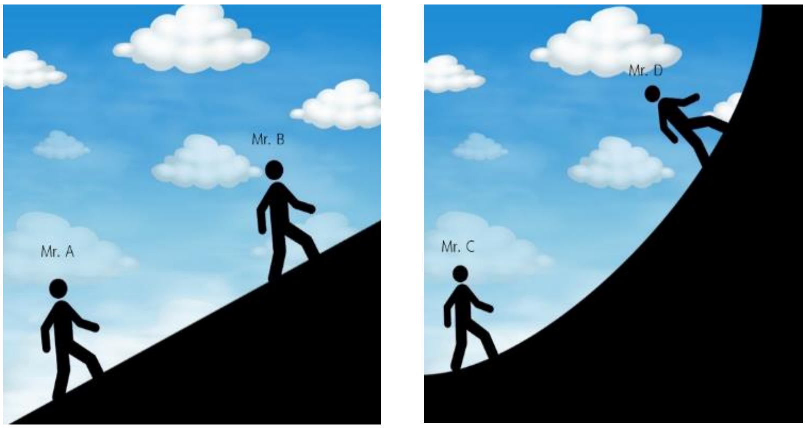

Figure 7 The scenario of moving up a steep hill

During the Concrete stage, we introduced real-world scenarios to help students connect with the concept of the slope of a curve at a given point, forming the basis of the Concrete stage in the CRA approach. Referring to Figure 7, we posed a scaffold question: “If Mr. A and Mr. B Walk at the same distance up the slope, starting from the positions shown in the image, do you think they would experience the same level of difficulty climbing? Why or why not?”. This question helps students understand that the slope of a straight line is constant at every point. On the other hand, for a curved steep hill, we used a scaffold question: “If Mr. C and Mr. D Walk at the same distance up the slope, starting from the positions shown in the image, do you think they would experience the same level of difficulty climbing? Why or why not?”. Students should independently understand that the slope of a curved line varies at each point. The main question in this lesson is: “How can we determine the slope of a curve at a given point?”

To determine the slope of the curve, Figure 8 will prompt students to consider which colored line most closely resembles the slope of the curve at each point. At this point in the lesson, students should be able to answer the question on their own: The slope of the curve at the given point is equal to the slope of the tangent line to the curve at that point.

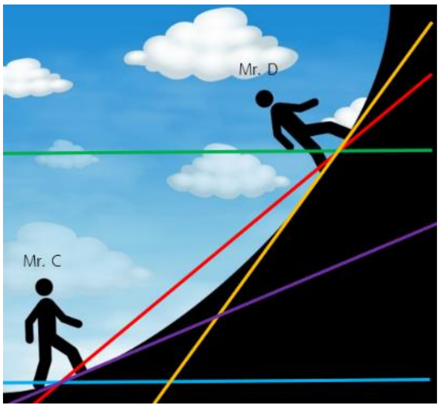

Figure 8 The scenario of moving up a steep hill with a segment line and tangent line at each point

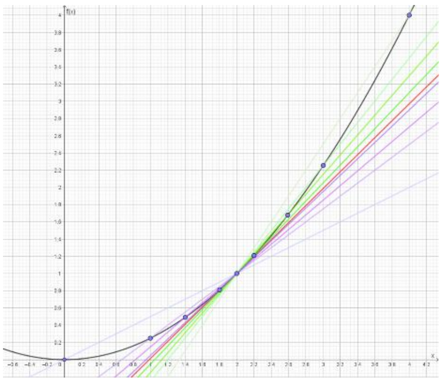

Figure 9 Graphical representation for slope of a curve

In the Representational stage, our goal was to guide students in learning how to find the slope of the tangent line at a point (which represents the slope of the curve at that point) using graphical tools, such as the Desmos program (https://www.desmos.com/calculator/pkhg1lobd9). In this stage, we also used questions to guide students in developing an understanding of how to find the slope of a curve. For example: “If you want to find the slope of a curve, what information do you think you need to know?”; “For a tangent line, we know only the point of tangency between the tangent line and the curve. How could we find another point to determine the slope of the tangent line?”; “For curve-secant lines, if we choose a line that intersects the curve at two points, the first point is where we are interested in finding the slope of the curve while the second point is very close to the first. What will the intersection of the curve look like in this case?”. Scaffolding questions, along with the graph, could prompt students to consider the scenario where a line intersects the curve at two points: the first point, where we are interested in finding the slope of the curve, and the second point, which is very close to the first. This secant line can have a slope that approaches the slope of the tangent line at the point of interest on the curve.

Finally, in the Abstract stage, we aimed to help students connect the concept of choosing a point close to a tangent point with mathematical symbols. The question that can help students is: "Which point on the curve $f(x)$ is very close to the point $(a, f(a))$?". This question aimed to help students understand that the point $(a+h, f(a+h))$, as h approaches 0, is very close to the point $(a, f(a))$. Finally, the slope of the tangent line to a curve at a given point (which represents the slope of the curve at that point) can be found using the concept of a limit, expressed as $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$.

Understanding calculus concepts necessitates the use of visual aids and mathematical representations to grasp its abstract nature. Consequently, lesson plans which were designed by using the CRA approach and scaffolded questions should bridge tangible, real-world contexts with the abstract mathematics of calculus through the careful selection of appropriate mathematical tools. Additionally, incorporating effective questioning techniques could enable students to explore and grasp concepts independently, with the teacher serving as a guide.

4. Results and Discussion

A Likert scale ranging from 1 to 5 was used in each lesson plan assessment form. An average score between 1.00 and 2.00 is considered "Needs Improvement", 2.01 to 3.00 is considered "Fair", 3.01 to 4.00 is considered "Good", and 4.01 to 5.00 is considered "Very Good". Results of the assessment of all four lesson plans, assessed by the three experts, were presented in Table 2.

Table 2 Results of Assessment of Lesson Plans Quality

Assessment List	Quality Level			Statistics	
	Expert 1	Expert 2	Expert 3	$\bar{x}$	S.D.
Lesson Plan 1: Limit of function					
Total (110 marks)	105	101	100	102	2.16
Mean	4.77	4.59	4.55	4.64	0.10
Lesson Plan 2: Continuity of function at point					
Total (110 marks)	100	99	103	100.67	1.70
Mean	4.55	4.50	4.68	4.58	0.08
Lesson Plan 3: Continuity of function on interval					
Total (110 marks)	97	102	98	99	2.16
Mean	4.41	4.64	4.45	4.50	0.10
Lesson Plan 4: Slope of a curve					
Total (110 marks)	101	101	102	101.33	0.47
Mean	4.59	4.59	4.64	4.61	0.02

The assessment results from the experts indicated that the average scores assigned by the three experts ranged from 4.00 to 5.00 across all four lesson plans. These results indicated that the four lesson plans were of very good quality. Additionally, the experts provided three key suggestions:

1) Clearly defining lesson objectives that can be assessed.
2) Reviewing lesson plans for spelling errors and clarity.
3) Strengthening the connection between the Concrete and Representational stages.

Based on the experts' suggestions, the lesson plans were revised and further developed to address various aspects. The first issue involves revising the learning objectives to ensure they are measurable and evaluable. According to Fadoli (2022), learning objectives play a central role in the teaching and learning process. Their inclusion in a lesson plan ensures that teachers remain on track and stay focused on the intended goals. If a teacher deviates from the objectives, these learning objectives can serve as a reminder to the teacher to realign with the lesson's purpose. Therefore, carefully formulating learning objectives enhances the likelihood of achieving meaningful learning outcomes. Additionally, teachers should thoughtfully select observable and operational verbs, as these helps define the cognitive domain and indicate the level of students' critical thinking. For example, the objectives of the lesson plans in this study were adjusted from statements like “Students understand...” to “Students can find...” to facilitate clear assessment and evaluation.

The next suggestion of the experts involves correcting errors and refining the use of mathematical symbols in the lesson plans. These aspects were thoroughly reviewed and revised to ensure accuracy and clarity.

Lastly, the experts recommended strengthening the connection between the Concrete and Representational stages. According to Fyfe and Nathan (2019), to enhance learning and facilitate the transfer of abstract concepts, modern theories emphasize the importance of explicitly linking concrete representations to the abstract ideas they symbolize. Concreteness fading is an instructional approach designed to establish these connections. Initially developed as a three-phase process, it begins with a tangible, physical representation of a concept, transitions to a visual or iconic depiction, and gradually progresses to a fully abstract representation. According to Fyfe et.al. (2014), the importance of concreteness fading which involves a transition process from a physical representation of a concept to a visual or iconic depiction, can conclude as follows: 1) aiding learners in interpreting ambiguous or complex abstract symbols by relating them to familiar concrete objects, 2) offering embodied perceptual and physical experiences that support abstract thinking, 3) helping learners develop a mental repository of memorable images to reference when abstract symbols become difficult to comprehend, and 4) guiding learners in identifying and retaining essential, generalizable properties while eliminating irrelevant concrete details.

Moreover, scaffolding techniques play a pivotal role in facilitating students' transition from the Concrete stage to the Representational stage within the Concrete-Representational-Abstract (CRA) instructional framework. By providing structured support, scaffolding helps students internalize mathematical concepts, enabling them to move from manipulating tangible objects to creating visual representations. This progression enhances their conceptual understanding and problem-solving skills.

The development of the lesson plans in this research aims to strengthen the connections between the Concrete and Representational stages. Specific improvements have been made to each lesson plan, as outlined below.

In the limits of functions lesson plan, the scenario of a diver’s path in the Concrete stage was modified to more closely resemble the graph of a function in the Representational stage. This adjustment was made to enhance the coherence and alignment between these two stages of learning.

In the continuity of function lesson plan, the curve section cards used in the Concrete stage were refined to better align with and resemble the graph of a function in the Representational stage. This improvement was made to strengthen the connection and enhance the similarity between the Concrete and Representational stages.

In the slope of a curve lesson plan, the connections between the three stages of the CRA approach were enhanced. Initially, in the Concrete stage, the slopes between two different points were compared, while in the Representational and Abstract stages, only a single slope was examined. To strengthen the coherence among the three stages, the experts recommended incorporating the exploration of both slopes in the Representational and Abstract stages as well.

The Concrete-Representational-Abstract (CRA) approach, combined with scaffolding techniques, has demonstrated significant advantages over Traditional-Based Learning (TBL), Inquiry-Based Learning (IBL), and Problem-Based Learning (PBL) in teaching calculus concepts. TBL often emphasizes procedural proficiency, which can lead to a superficial understanding of complex mathematical ideas (Khalaf & Zin, 2018). IBL and PBL promote deeper comprehension through exploration and problem-solving but may lack the structured support necessary for students struggling with abstract concepts (Liang, 2022; Pogorelova, 2023). Moreover, according to Tarmizi (2010), one of the major challenges students face in learning calculus is the difficulty in visualizing and comprehending abstract mathematical concepts. Therefore, connecting mathematical representations to abstract mathematics is essential for students' learning in calculus. The CRA approach provides a systematic progression from concrete manipulatives to mathematical representations and finally to abstract symbols, effectively bridging the gap between tangible experiences and abstract reasoning. It is, therefore, an appropriate teaching method for calculus. Furthermore, a meta-analytic review highlighted the efficacy of the CRA approach as a math intervention, underscoring its effectiveness in enhancing mathematical outcomes (Ebner et al., 2024). Collectively, these studies emphasize the superiority of the CRA approach combined with scaffolding over TBL, IBL, and PBL methodologies in fostering a strong mathematical understanding in calculus education.

Scaffolding techniques play a crucial role in enhancing students' understanding of calculus when integrated with the Concrete-Representational-Abstract (CRA) approach. By providing structured, step-by-step guidance, scaffolding helps bridge the gap between students' prior knowledge and new mathematical concepts, facilitating a smoother transition from concrete experiences to abstract reasoning. Scaffolding, through methods such as guided questioning, stepwise problem breakdown, and visual representations, enables learners to develop a deeper comprehension of these complex ideas (Anda & Aman, 2022; Frederick & Courtney, 2014; Dy & Lapinid, 2023). Additionally, scaffolding techniques help reduce cognitive overload by segmenting complex calculus problems into manageable steps, allowing students to engage with the content without feeling overwhelmed. These findings highlight the effectiveness of combining scaffolding with the CRA approach in calculus education, ensuring students receive the necessary support to build a strong foundation in mathematical reasoning and problem-solving.

5. Conclusion

In this study, the researcher designed a calculus lesson plan integrating the Concrete-Representational-Abstract (CRA) approach with scaffolding techniques. The researcher designed and developed a total of four lesson plans covering four topics of calculus for high school students: the limit of a function, the continuity of a function at points, the continuity of a function on intervals, and the slope of a curve.

Each lesson plan was structured following the CRA approach combined with scaffolding techniques. In Concrete stage, real-life scenarios, such as a diver's underwater exploration or moving up a slope, were introduced to help students visualize and comprehend concepts more effectively. This step also incorporated card games to engage students, promote enjoyment, and facilitate hands-on learning with tangible objects. In Representational stage, the use of graphical representations, particularly graphs of a function, was emphasized to bridge the gap between concrete experiences and abstract understanding, which is crucial in learning calculus (Berry & Nyman, 2003). Finally, in the Abstract stage, students were guided to explore mathematical symbols relevant to calculus, such as limit of function notation. This step also conveyed key concepts, including continuity and calculating the slope of a curve, fostering a deeper understanding of abstract mathematical ideas. Furthermore, scaffolding questions were incorporated into every step of the CRA approach, serving as essential tools to guide students' thinking and direct their learning in alignment with the teaching objectives (Ministry of Education, 2012). Scaffolding also played a pivotal role in enabling students to independently discover new concepts, with the teacher acting as a facilitator rather than simply delivering information, as is common in traditional instruction (Wood et al., 1976).

Each lesson plan was assessed for quality by three experts using a Likert scale-based assessment form ranging from 1 to 5. The assessment results indicated that all four plans achieved average scores ranging from 4.00 to 5.00, demonstrating their "Very good" quality.

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